# $\mathbb{E}$xercise sheet 9: Harmonic functions and Brownian motion

**Exercise 1** — *Harmonic functions and martingales.*
Let $D$ be a bounded domain satisfying the Poincaré cone condition and $T$ its exit time. Let $h : \overline{D} \to \mathbb{R}$ be continuous and harmonic inside $D$. Show that for $x \in D$, under $\mathbb{P}_x$, the process $t \mapsto h(B_{t\wedge T})$ is a closed martingale.
Conversely show that if $h$ is defined on some domain $U$ with the property that for every $\overline{B}(x,\epsilon) \subset U$, $t \mapsto h(B_{t\wedge T_{\partial B(x,\epsilon)}})$ is a martingale under $\mathbb{P}_x$, then $h$ is harmonic.

**Exercise 2** — *A lemma for the Poincaré cone condition.*
Let $C$ be an open cone based in 0. We wish to show that the function $\phi(x) = \mathbb{P}_x(T_{\partial B(0,1)} < T_{\partial C})$ is bounded away from 1 on $\overline{B}(0,1/2) \setminus C$.

(1) Why can't we use the maximum principle for $\phi$ on $\overline{B}(0,1/2) \setminus C$ ?
(2) Bound $\phi$ by some (similarly defined) function to which the maximum principle can be applied, and conclude.

**Exercise 3** — *Counterexample.*
Let $D = B(0,1) \setminus 0 \subset \mathbb{R}^2$ and consider the Laplace equation $\Delta u = 0$ with Dirichlet boundary conditions $u(0) = 0$ and $u(x) = 1$ for $x \in \partial B(0,1)$. Show that the Brownian expectation does not define a solution. Show that there can't exist a solution (you may use the fact that if $u(x) = g(|x|)$, then $\Delta u(x) = g''(|x|) + \frac{1}{x} g'(|x|)$.)

**Exercise 4** — *The binary splitting martingale.*
Let $X$ be centered with finite variance and $(X_n)_n$ be the associated binary splitting martingale, defined as follows: Let $\mathcal{G}_0$ the trivial $\sigma$-field, and for $n \geq 0$, set $X_n = \mathbb{E}[X \mid \mathcal{G}_n]$, $\xi_n = \operatorname{sgn}(X - X_n)$ and $\mathcal{G}_{n+1} = \sigma(\xi_0, \ldots, \xi_n)$. You know that $(X_n)_n$ is a martingale for the filtration $(\mathcal{G}_n)_n$, that it is bounded in $L^2$ hence converges a.s. and $L^1$ to some random variable $X_\infty$. We still need to show that $X_\infty = X$ a.s.

(1) Express $X_{n+1} - X_n$ so that its positive and negative part are explicit. Use this to compute $|X_{n+1} - X_n|$.
(2) Deduce that $|X_n - X|$ goes to 0 in $L^1$ and conclude.