$\mathbb{E}$xercise sheet 4: stopping times and Markov property (v3)

Exercise 1 — *Stopping times.*
Let (T_n) be a sequence of stopping times for a filtration $\mathcal{F}$. We call $\mathcal{F}^+$ the right-continuous version of $\mathcal{F}$.

(1) Assume that for every ω, the sequence $(T_n(\omega))$ is increasing with limit $T(\omega)$. Show that T is a $\mathcal{F}$-stopping time.
(2) Assume that for every ω, the sequence $(T_n(\omega))$ is decreasing with limit $T(\omega)$. Show that T is a $\mathcal{F}^+$-stopping time.
(3) Assume that for every ω, the sequence $(T_n(\omega))$ is decreasing and is eventually equal to $T(\omega)$. Show that T is a $\mathcal{F}$-stopping time.

Exercise 2 — *Measurability of the stopped process.*
Let B be a $\mathcal{F}$-Brownian motion and T a $\mathcal{F}$-stopping time. Show that $(T, B_{\min(t,T)})_{t\geq 0}$ is $\mathcal{F}_T$-measurable.

Exercise 3 — *Counter-example.*
Show that the first hitting time by B of the maximum of B on $[0,1]$ is not a stopping time.

Exercise 4 — *Brownian motion on the circle.*
Define a Brownian motion on the circle $\mathbb{S}^1$ by setting $X_t = e^{iB_t}$ for $t \geq 0$. What is the distribution of the last point hit by X in $\mathbb{S}^1$?

Exercise 5 — *Hitting time.*
Compute the distribution of a hitting time T_a of some level $a > 0$.

Exercise 6 — *A bit more on differentiability.*
We know that almost surely, B is nowhere differentiable. Set $D^*B(t) = \limsup_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$ and $D_*B(t) = \liminf_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$.

(1) Show that B almost surely not bounded above nor below. Deduce that $D^*B(0) = +\infty$ a.s. and $D_*B(0) = -\infty$ a.s.
(2) Deduce that almost surely, the Lebesgue measure of times t such that $D^*B(t) \neq +\infty$ or $D_*B(t) \neq -\infty$ is 0.
(3) Show that with probability one a fixed point t is not a one-sided local maximum of B. Deduce that with probability one there exists a density of exceptional random times where $D^*B(t) \leq 0$.
(4) Show that there almost surely exists an uncountable density of points t where $D^*(t) = 0$. (Hint : consider $\tau(x) = \inf\{t \geq 0, B_t = x\}$. Show that this is almost

surely a strictly increasing function whose discontinuity points are dense and deduce that $V_n = \{x \geq 0, \exists h \in (0, 1/n), \tau(x-h) < \tau(x) - nh\}$ is open and dense. What can be said about $\bigcap_{n \geq 1} V_n$?)

Exercise 7 — *The set of zeros of B is perfect.*
Let B be a Brownian motion, and $Z = \{t \geq 0 : \; B_t = 0\}$. Show that almost surely, Z is a closed set without isolated points.