Solutions for $\mathbb{E}$xercise sheet 3 : Regularity properties & miscellanea

Solution 1 — *Boring but important measure theoretic stuff.* (1) By definition, $\mathcal{B}(\mathbb{R})^{\otimes I}$ is the smallest σ-algebra that makes all projections $p_i : x \mapsto x_i$ measurable, so it is generated by the 1-dimensional cylinder sets $\{x \in \mathbb{R}^I, x_i \in A\}$ for $A \in \mathcal{B}(\mathbb{R})$ and $i \in I$. A π-system that generates is that of the finite-dimensional cylinder sets $\{x \in \mathbb{R}^I, x_{i_1} \in A_1, \dots x_{i_n} \in A_n\}$ for $n \geq 0$, $i_1, \dots, i_n \in I$, $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. The fact that a probability measure on $\mathcal{B}(\mathbb{R})^{\otimes I}$ is determined by the f.d.m's is then a direct application of the π-λ theorem.

(2) $\Longrightarrow$ because projections are measurable, $\Longleftarrow$ because the preimage of a 1-dimensional cylinder set is an event of the form $\{X_i \in A\}$ for some i and Borel set A.

(3) $B(\mathbb{R})^{\otimes I} \subset B(\mathbb{R}^I)$ because the latter makes all continuous functions measurable, the projections are continuous, and the former is the smallest one that makes all projections continuous. The other way around is as follows: a basis for the topology of $\mathbb{R}^I$ is the set of finite-dimensional open cylinders $\{x \in \mathbb{R}^I, x_{i_1} \in O_1, \dots x_{i_n} \in O_n\}$. Any open set is an union of such basis elements, which can be rewritten as a countable union thanks to separability of R^I. Then finite-dimensional open cylinders, which belong to $\mathcal{B}(\mathbb{R})^{\otimes I}$, have all open sets in their generated σ-algebra, hence all of $B(\mathbb{R}^I)$. This shows $B(\mathbb{R}^I) \subset B(\mathbb{R})^{\otimes I}$.

(4) The inclusion comes from the previous question. Let us show that $\mathcal{B}(\mathbb{R})^{\otimes I}$ is included in the following σ-algebra: the family of subsets of $\mathbb{R}^I$ of the form $\{x \in \mathbb{R}^I : (x_j)_{j \in J} \in B\}$ for some countable subfamily $J \subset I$ and some $B \in \mathcal{B}(\mathbb{R})^J$. This indeeds forms a σ-algebra $\mathcal{C}$, that contains all one-dimensional cylinders, hence the whole of $\mathcal{B}(\mathbb{R})^{\otimes I}$. But an open set of the form $\{x \in I : \exists i \in I, x_i > 0\}$ cannot belong to $\mathcal{C}$. Hence the strict inclusion.

(5) $(\mathcal{B}(\mathbb{R})^{\otimes [0,1]})_{|\mathcal{C}([0,1])} \supset \mathcal{B}(\mathcal{C}([0,1])$ because projections are continuous w.r.t. the topology of $\mathcal{C}([0,1])$. For the other way around it suffices to show that the distance on $\mathcal{C}([0,1])$ is measurable w.r.t. $(\mathcal{B}(\mathbb{R})^{\otimes [0,1]})_{|\mathcal{C}([0,1])}$ because then an open ball can be rewritten as $d(x, \cdot)^{-1}([0,l))$, hence is measurable. But we have

$$d(x, y) = \sup_{0 \leq t \leq 1, t \in \mathbb{Q}} |x(t) - y(t)|.$$

which immediately gives measurability of d.

Solution 2 — *Indistinguishability and modifications.* (1) Indistinguishable $\Longrightarrow$ Modification

(2) By countable union with probability one they are equal over the rationals of I. Hence they are equal everywhere.

(3) Show that "X is indistinguishable from a Brownian motion" is equivalent to "X is a $\mathcal{C}(\mathbb{R}_+)$-valued random variable (up to a.s. equality) which is distributed like a Brownian motion". $\Longrightarrow$ X is a.s. equal to a Brownian motion B. Let us check that B is indeed a random variable in the space $(\mathcal{C}(\mathbb{R}_+), \mathcal{B}(\mathcal{C}(\mathbb{R}_+))) = (\mathcal{C}(\mathbb{R}+), (\mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}^+})_{|\mathcal{C}(\mathbb{R}^+)})$. This comes immediately from question 2 of the previous exercise, since for every t, B_t is a random variable. Now of course B has the law of a Brownian motion, since it's a Brownian motion...

$\Longleftarrow$ Here X is a.s. equal to some $\mathcal{C}(\mathbb{R}^+)$-valued r.v. B that has the law of the Brownian motion. Let us check that it is a Brownian motion: for all ω, B is in $\mathcal{C}(\mathbb{R}^+)$ so the paths are continous. B_t is indeed a random variable because of question 2 of the previous exercise, and the law of the finite-dimensional marginals are indentified because of question 1 of the previous exercise.

Solution 3 — *Local regularity and long-term behavior.* (1) Immediate since almost surely $X_t = o(1)$ as $t \to 0$

(2) We know (lecture !) that almost surely, X is not locally $1/2+\epsilon$-Hölder at 0. So $\limsup_{t\to\infty} \frac{X_{1/t}}{1/t^{1/2+\epsilon}} = \infty$ almost surely, which gives the claim after rewriting. Also (lecture !) there exists C such that $|X(h) - X(0)| \leq m_X([0,1], h) \leq C\sqrt{h \log(1/h)}$ when h is small enough. Hence the claim when taking $h = 1/t$, $t \to \infty$.

(3) (a) By Fatou's lemma,

$$\mathbb{P}(\limsup_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}} < c) \leq \mathbb{P}(\liminf_{n\to\infty}\{B_{2^{-n}} < c\sqrt{2^{-n}}\})$$
$$\leq \liminf_{n\to\infty} \mathbb{P}(B_{2^{-n}} < c\sqrt{2^{-n}}) = \liminf_{n\to\infty} \mathbb{P}(B_1 \leq c) < 1.$$

(b) Lévy's construction tells us that $B_{2^{-n}} = 2^{-n}N_0 + \sum_{k=0}^{n-1} 2^{-n+k/2} N_{0,k}$. Hence

$$\frac{B_{2^{-n}}}{\sqrt{2^{-n}}} = 2^{-n/2}N_0 + \sum_{k=0}^{n-1} 2^{(k-n)/2} N_{0,k}$$

Each fixed term of the sum goes to 0 separately. So the lim inf does not change when the first few terms are removed. We deduce that $\liminf_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}}$ is measurable w.r.t. the σ-algebra $\sigma(N_{0,k}, k \geq K)$ for all fixed K, thus to the tail σ-algebra.

(c) $\{\limsup_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}} < c\}$ is a tail event for a sequence of independent random variables, hence by Kolmogorov's 0-1 law it has probability 0 or 1, and it is not 1 because of question 3a. Hence with probability one $\limsup_{t\to 0} B_t/\sqrt{t} \geq \limsup_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}} = \infty$. So B is not locally Hölder at 0 and at infinity, $\limsup B_t/\sqrt{t} = +\infty$ by time-reversal. Since $B \stackrel{d}{=} -B$, it comes that $\liminf B_t/\sqrt{t} = -\infty$ too. We get for free that B is almost surely surjective.

Solution 4 — *Quadratic and absolute variation.* (1) We first try to guess what the L^2 limit could be. If we have L^2 convergence to say X, since L^2 convergence implies L^1 convergence, we get that $\mathbb{E}[X] = \lim_k \mathbb{E}[\sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2] = \lim_k t = t$ as we observe a telescoping series. So $\mathbb{E}[X] = t$ and if by any chance X was deterministic[1], we'd have $X = t$. Let's try to show convergence to t. We rewrite

$$A_k = \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 - t = \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 - (t_i^{(k)} - t_{i-1}^{(k)}).$$

Then

$$\begin{aligned}\mathbb{E}[A_k^2] = \mathrm{Var}(A_k) &= \sum_{i=1}^{\#\underline{t}^{(k)}} \mathrm{Var}((B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 - (t_i^{(k)} - t_{i-1}^{(k)})) \\ &= \sum_{i=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2 \,\mathrm{Var}(Z^2 - 1) = (\mathrm{cst}) \sum_{i=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2. \quad \leq (\mathrm{cst})\, t|\underline{t}^{(k)}|\end{aligned}$$

which goes to 0. We have moved freely between variance and second moment because everything is centered, used the independence of increments and the decomposition of variance over an independent sum, the fact that a standard Gaussian Z has a fourth moment, and Hölder$(1,\infty)$ at the end. We have $A_k \to 0$ in L^2 which is the claim.

(2) If $(\underline{t}^{(k)})_k$ is such that $\sum_{k=1}^{\infty} \sum_{j=1}^{\#|\underline{t}|} (t_i - t_{i-1})^2 < \infty$, then we get $\sum_{k=1\infty} \mathbb{E}[A_k^2] < \infty$. For $\epsilon \in \mathbb{Q}_+^*$ we have $\mathbb{P}(|A_k| > \epsilon) \leq \epsilon - 2\,\mathbb{E}[A_k^2]$ which is summable. So by Borel-Cantelli, almost surely, for large n, $|A_k| < \epsilon$. We invert " $\forall \epsilon \in \mathbb{Q}_+^*$" and "almost surely" by countable union and we're done.

(3) If B has bounded variation, then it is not hard to show that the quadratic variation is 0 (once again Hölder$(1,\infty)$). But this is a.s. impossible because of the previous question.

Solution 5 — *The precise constant (Lévy, 1937).* (1) The upper bound comes from the inequality $\int_x^\infty e^{-t^2/2} dt \leq \int_x^\infty \frac{t}{x} e^{-t^2/2} dt$. The lower bound can be obtained by differentiating the difference.

[1]Reasonable because it seems that there could be 0-1 law w.r.t. Lévy construction

(2) First of all, $\mathbb{P}(E_{k,n}) = \mathbb{P}(B_{(k+1)2^{-n}} - B_{k2^{-n}} \geq c\sqrt{2^{-n}\log(2^n)}) = \mathbb{P}(B_1 \geq c\sqrt{n\log 2}) \geq \frac{1}{1000c\sqrt{n}} 2^{-c^2 n/2}$. Then

$$\mathbb{P}(\forall 0 \leq k \leq 2^- n, B_{(k+1)2^{-n}} - B_{k2^{-n}} < c\sqrt{2^{-n}\log(2^n)}) = \mathbb{P}(\bigcap_k E_{k,n}^{\complement})$$
$$= \prod_k (1 - \mathbb{P}(E_{k,n})) \leq (1 - \frac{1}{1000c\sqrt{n}} 2^{-c^2 n/2})^{2^n} \leq \exp(-2^n \frac{1}{1000c\sqrt{n}} 2^{-c^2 n/2})$$
$$= \exp(-\frac{1}{1000c\sqrt{n}} 2^{(1-c^2/2)n}) = \text{summable in} n.$$

So by Borel-Cantelli, we get that infinitely often in n, there is an increment of length 2^{-n} that exceeds $c\sqrt{2^{-n}\log(2^n)}$. This implies the claim.

(3)

$$\mathbb{P}(\exists [s,t] \in \Lambda_n(m), |B(t) - B(s)| > c\sqrt{|t-s|\log(1/|t-s|)})$$
$$\leq m2^{n/m}\,\mathbb{P}(|B_1| \geq c\sqrt{n/m \log 2})$$
$$\leq m2^{n/m} \frac{1}{\sqrt{2\pi}c\sqrt{n/m\log 2}} 2^{-(c^2/2)n/m} = \text{summable in } n$$

So almost surely, for n large enough, any interval in $\Lambda_n(m)$ has the required growth bound.

(4) Take m to be determined later in terms of ϵ. Then given t and s, we can find n so that $1 \leq |t-s|/2^{-n/m} \leq (2^{1/m}) \leq 1 + \epsilon/3$. We can now find k so that $|s - \frac{k}{m}2^{-n/m}| \leq \frac{1}{m}2^{-n/m} \leq \frac{1}{m}|t-s|$. Set $s' = \frac{k}{m}2^{-n/m}$, $t' = (\frac{k}{m}+1)2^{-n/m}$. Then $|s'-s| \leq \frac{1}{m}|t-s|$ and $|t'-t| \leq |t'-s'| + |s'-s| \leq (2^{1/m}-1)|t-s| + \frac{1}{m}|t-s|$. Now choose retrospectively m so that $2^{1/m} - 1 + \frac{1}{m} < \epsilon$ and $\frac{1}{m} < \epsilon$ makes everything work. Remark that we additionaly get $|t'-s'| \leq |t-s|$ which eases the solution of the next question.

(5) Fix ϵ and m accordingly. Now almost surely, there is n_0 such that for $n \geq n_0$, all intervals in $\Lambda_n(m)$ have the growth bound with the constant c. Moreover, from the lecture, almost surely there is a h_0 such that all intervals of length $< h_0$ have the growth bound with the constant C from the lecture. Now take s,t such that $|s-t| \leq \epsilon$, $\epsilon|s-t| \leq h_0$ and $|s-t| \leq 2^{-n_0/m}$. Then consider s', t' as in the previous question. It comes that $|t'-t|, |s'-s| \leq h_0$ and that $|s'-t'| \in \Lambda_n(m)$ with $n \geq n_0$. Hence

$$|B_t - B_s| \leq |B_t - B_t'| + |B_s - B_s'| + |B_t - B_s|$$
$$\leq C\sqrt{|s'-s|\log(1/|s'-s|)} + C\sqrt{|t'-t|\log(1/|t'-t|)} + c\sqrt{|t'-s'|\log(1/|t'-s'|)}$$
$$\leq 2C\sqrt{\epsilon|t-s|\log(1/(\epsilon|t-s|))} + c\sqrt{|t-s|\log(1/|t-s|)}$$
$$\leq (2C\sqrt{\epsilon(1+1)} + c)\sqrt{|t-s|\log(1/|t-s|)}.$$

Where at the second inequality we used the increasing character (close to 0) of $x \mapsto \sqrt{x \log(1/x)}$ and at the last one we used $\log(1/\epsilon) \leq \log(1/|s-t|)$. The constant obtained can be brought arbitrarily close to $\sqrt{2}$ as c was arbitrary $> \sqrt{2}$, ϵ arbitrary > 0 and C fixed.